

Cosmological Equations and Thermodynamics on Apparent Horizon in Thick Braneworld

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Abstract

We derive the generalized Friedmann equation governing the cosmological evolution inside the thick brane model in the presence of two curvature correction terms: a four-dimensional scalar curvature from induced gravity on the brane, and a five-dimensional Gauss-Bonnet curvature term. We find two effective four-dimensional reductions of the generalized Friedmann equation in some limits and demonstrate that the reductions but not the generalized Friedmann equation can be rewritten as the first law of equilibrium thermodynamics on the apparent horizon of thick braneworld.

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I. INTRODUCTION

Inspired by black hole thermodynamics, a profound connection between gravity and thermodynamics has been argued to exist. In [1] Jacobson first showed that the Einstein gravity can be derived from the first law of thermodynamics in the Rindler spacetime. For a general static spherically symmetric spacetime, Padmanabhan pointed out that Einstein equations at the horizon give rise to the first law of thermodynamics [2]. Recently the study on the connection between gravity and thermodynamics has been extended to cosmological context. Frolov and Kofman in [3] employed the approach proposed by Jacobson [1] to a quasi-de Sitter geometry of inflationary universe, and they calculated the energy flux of a background slow-roll scalar through the quasi-de Sitter apparent horizon. By applying the first law of thermodynamics to a cosmological horizon, Danielsson obtained the Friedmann equation in the expanding universe [4]. In the quintessence dominated accelerating universe, Bousso [5] showed that the first law of thermodynamics holds at the apparent horizon. Cai and Kim [6] generalized the derivation of the Friedmann equations from the first law of thermodynamics to the spacetime with any spatial curvature. This study has also been generalized to the $f(R)$ gravity [7, 8] and scalar-tensor gravity theory [9]. It has been disclosed that the first law of thermodynamics can not be constructed unless introducing the non-equilibrium entropy production term.

Besides gravity theories in four dimensions, the study on the connection between gravity and thermodynamics has also been extended to the braneworld cosmology [10, 11, 12]. The main merit of braneworld scenario is that it provided a novel approach to resolve the cosmological constant and the hierarchy problems [13, 14]. It has been found that the first law of thermodynamics on the apparent horizon can be derived from the Friedmann equations in the Randall-Sundrum (RS) braneworld [10, 11], and also in braneworld with curvature corrections including the five-dimensional (5D) Gauss-Bonnet (GB) curvature correction [12] and the four-dimensional (4D) scalar curvature from induced gravity on the brane [11]. The former correction is inspired by superstring theory which suggests the GB curvature correction as the first and dominant quantum corrections to the Einstein-Hilbert action for a ghost-free theory [15]. The combined action in five dimensions gives the most general action with second-order field equation, as shown by Lovelock [16]. Their cosmological effects have been discussed in several papers [17]. The Dvali-Gabadadze-Porati

(DGP) model suggests the second curvature correction term to RS model, the 4D scalar curvature term. This induced gravity correction term can be interpreted as arising from a quantum effect due to the interaction between the bulk gravitons and the matter on the brane [18]. The DGP scenario with GB correction was also given, where the well-known DGP feature of late-time acceleration without dark energy is preserved, but there is an intriguing and new feature that the singularity in the early universe may be removed [19, 20]. In these braneworld scenarios the exact black hole solution has not been found until now, and it was pointed out that the connection between gravity and thermodynamics can shed lights on the entropy of the braneworld [11, 12].

Although so many gravity theories have been linked to thermodynamics, it is still unclear whether the connection always exists in arbitrary gravity theories. Validating the connection in a more general gravity theory may be helpful to answer the question. On the other hand, it is worth to note that the previous mentioned braneworld scenarios are restricted by a simplifying assumption that the brane is infinitely thin along the extra dimension with solitonic localized matter distributions, and in more realistic models the thickness of the brane should be taken into account. Study of thick branes in the string inspired context of cosmology began almost simultaneously with the study of thin brane. Different approaches were used to define and handle the thickness. Most papers described the brane as the domain wall, based on gravity coupled to scalars fields [21]. A smoothing or smearing mechanism was used in [22] to modify the RS ansatz. Authors in [23] introduced a thickness to the brane by smoothing out the warp factor of a thin braneworld to investigate the stability of a thick brane. Mounaix and Langlois proposed a general approach to derive generalized Friedmann equation, where the 4D effective brane quantities are obtained by integrating the corresponding 5D ones along the extra dimension over the brane thickness [24]. In the case of a RS type cosmology, these quantities are reduced to yield the cosmological equations in the low energy limit and in the limit for a small brane thickness. When the brane thickness is not small enough, it was disclosed that no effective 4D reduction is possible unless some auxiliary quantities are introduced. Other methods have also been presented recently, based on gluing a thick brane, considered as a regular manifold, to two different manifolds on both sides of it [25], and integrating the 5D Einstein equations along the fifth dimension, while neglecting the parallel derivatives of the metric in comparison with the transverse ones [26]. It is interesting to note that the thickness effect has been considered to mimic the dark

energy by recent observations [27].

In this paper, following Mounaix and Langlois's approach to define the 4D effective brane quantities, we will study whether the thermodynamics may relate to gravity on thick brane. To make our results with more generality, we will not restrict us on the RS thick brane, instead we will generalize the thick brane model in the presence of two curvature corrections: the 5D GB curvature term and the 4D scalar curvature from induced gravity on the brane. Compared with the investigation on the thermodynamics in other gravity theories [6, 9, 10, 11, 12], it should be stressed that this task is highly nontrivial. This is partially because we must obtain the exact Friedmann equations on thick brane with curvature corrections and partially because the 4D effective thermodynamical brane quantities have not been defined until now. We will derive the generalized cosmological equations and obtain their 4D effective reductions. So one can treat this work partially as the direct generalization of Mounaix and Langlois's one, which will give access to a description of the intermediate cases between the attractive DGP brane world with GB curvature correction (where brane thickness is infinitely thin) and the opposite limit of an infinitely thick brane, which effectively corresponds to the familiar 5D GB gravity (since the infinitely thick 4D induced scalar curvature may be absorbed into 5D Einstein-Hilbert scalar curvature). Furthermore, since our work will make the braneworld scenarios discussed on [10, 11, 12] more realistic and the gravity theory more general, we can expect, besides helping the investigation on black hole of more realistic braneworlds, to give further understanding on the deep connection between gravity and thermodynamics.

This paper is arranged as follows. In section 2, we will generalize the Friedmann equation of thick brane model with the 5D GB interaction and 4D scalar curvature term. In the following section, we will reduce the Friedmann equation in some limits. Then we will study the thermodynamical behavior of the reduced Friedmann equations in section 4. Our conclusion and discussion will be present in the last section.

II. COSMOLOGICAL EQUATIONS FOR THICK BRANEWORLDS

Let us consider a thick braneworld model. For convenience and without loss of generality we choose the extra dimension coordinates y such that the brane is center at $y = 0$ and localized between $y = -1/2$ and $y = 1/2$. For simplicity, the spacetime has been restricted

with Z_2 symmetry under the transformation $y \rightarrow -y$. It is then always possible to find a Gaussian coordinate system, starting from the hypersurface representing the center of the brane. We consider an action which incorporates the induced gravity and GB corrections

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \{ \mathcal{L}_{EH} + \alpha \mathcal{L}_{GB} \} + \frac{1}{2\kappa_4^2} \int_{-1/2}^{1/2} dy \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_{IG}, \quad (1)$$

where κ_5 (κ_4) is the bulk (brane) gravitational constant and g denotes the 5-dimensional bulk metric. The brane metric \tilde{g} is defined as follows. For each fixed $y = y_f$ between $y = -1/2$ and $y = 1/2$, the induced metric $\tilde{g}_{AB} \equiv g_{AB} - n_A n_B$, where vector n_A is normal to the slice of the brane at fixed y_f . $\mathcal{L}_{EH} = R - 2\Lambda$ is the 5D Einstein-Hilbert Lagrangian with negative cosmological constant $\Lambda < 0$. The GB curvature correction term \mathcal{L}_{GB} is

$$\mathcal{L}_{GB} = R^2 - 4R_{AB}R^{AB} + R_{ABCD}R^{ABCD}.$$

We can define the GB coupling α through string energy scale g_s as $\alpha = \frac{1}{8g_s^2}$. The second term of the action is the generalization of the induced gravity action describing thin brane $\frac{1}{2\kappa_4^2} \int \delta(y) dy \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_{IG}$ [18]. The induced gravity Lagrangian $\mathcal{L}_{IG} = \tilde{R} - 2\kappa_4^2 \lambda$ consists of 4D scale curvature \tilde{R} and brane tension $\lambda > 0$, noting λ is assumed as strictly constant in y . We can define the crossover length scale of induced gravity by $r_c = \kappa_5^2 / \kappa_4^2$. For convenience, we will choose the unit $\kappa_5 = 1$ throughout this paper. One can recover the thick RS model when $r_c = \alpha = 0$. The thick RS model with GB correction and the thick DGP model correspond to the case with $r_c = 0$ and $\alpha = 0$, respectively.

By varying the action in Eq. (1) with respect to the bulk metric, we obtain the field equation

$$G_{AB} + 2\alpha H_{AB} = T_{AB}|_{total}, \quad (2)$$

where H_{AB} is the second order Lovelock tensor [16]

$$H_{AB} = R R_{AB} - 2R_A^C R_{BC} - 2R^{CD} R_{ABCD} + R_A^{CDE} R_{BCDE} - \frac{1}{4} g_{AB} \mathcal{L}_{GB}.$$

The total energy-momentum tensor $T_{AB}|_{total}$ is decomposed into bulk and brane components

$$T_{AB}|_{total} = T_{AB}|_{bulk} + T_{AB}|_{brane}. \quad (3)$$

The bulk component is

$$T_{AB}|_{bulk} = -\Lambda g_{AB}. \quad (4)$$

Since we are interested in the cosmological behavior of the braneworld we take the metric ansatz

$$ds^2 = -n^2(t, y)dt^2 + a^2(t, y)\gamma_{ij}dx^i dx^j + r_b^2 dy^2 \quad (5)$$

where γ_{ij} is a three-dimensional maximally symmetric metric whose spatial curvature is characterized by $k = 0, \pm 1$. For simplicity, we consider the flat space $k = 0$ in present work. The energy-momentum tensor of the matter content in the brane is of the form

$$T_A^B|_{brane} = \text{diag} \left[\sqrt{\frac{\tilde{g}}{g}}(-\rho, p, p, p), p_T \right] - \sqrt{\frac{\tilde{g}}{g}} r_c \tilde{G}_A^B, \quad (6)$$

where the brane tensor λ is associated in

$$\rho \equiv \rho_m + \lambda, \quad (7)$$

\tilde{G}_A^B arises from the scalar curvature in Eq. (1), ρ, p, p_T are functions of t and y , and $\sqrt{\frac{\tilde{g}}{g}} = r_b$ is the brane thickness which is assumed to be time independent.

Our goal is to establish effective cosmological equations and study the corresponding thermodynamics for an observer living in the brane. Because of the finite thickness of the brane, there is some arbitrariness in the definition of what the effective 4D quantities should be. Here we adopt the prescription proposed by Mounaix and Langlois, in defining the 4D effective quantity $|Q|(t)$ associated to a 5D quantity $Q(t, y)$ as its spatial average over the brane thickness [24]

$$|Q|(t) \equiv \langle Q(t, y) \rangle,$$

where $\langle Q(t, y) \rangle = \int_{-1/2}^{1/2} Q(t, y) dy$. The 4D "observable" counterparts of a, ρ and p thus are

$$|a| \equiv \langle a \rangle, \quad |\rho| \equiv \langle \rho \rangle, \quad |p| \equiv \langle p \rangle,$$

and the observable Hubble parameter is

$$|H| \equiv \frac{\langle \dot{a} \rangle}{\langle a \rangle} = H_{\langle a \rangle}. \quad (8)$$

These observables are enough to establish effective cosmological equations. Later we will further give the 4D observable counterparts about thermodynamics.

Now, we study the components of field equations (2), with the metric (5) and energy-momentum tensors (3), (4) and (6). For later use, we define

$$\Phi = \frac{\dot{a}^2}{n^2 a^2} - \frac{a'^2}{r_b^2 a^2}, \quad (9)$$

where prime and dot denote the derivative with respect to y and t , respectively. Read 05 component of the field equations (2)

$$3\left(\frac{n'}{n}\frac{\dot{a}}{a} - \frac{\dot{a}'}{a}\right)(1 + 4\alpha\Phi) = 0$$

which yields

$$n(t, y) = \xi(t)\dot{a}(t, y),$$

where ξ depends on the normalization prescription for n . In following, we will take the normalization $\langle n \rangle = 1$ which gives

$$\xi = \langle \dot{a} \rangle^{-1}. \quad (10)$$

The 00 component of the field equations (2) then reads

$$(1 + 4\alpha\Phi)\frac{a''}{ar_b^2} = \Phi - \frac{1}{3}\left(\frac{\rho}{r_b} - \frac{3r_c}{r_b}\frac{\dot{a}^2}{n^2a^2} + \Lambda\right). \quad (11)$$

After inserting Φ (9), Eq. (11) can be rewritten as following two useful expressions, under the normalization (10),

$$(1 + 4\alpha\frac{\langle \dot{a} \rangle^2}{a^2})\frac{(a^2)''}{2} - \frac{4\alpha}{r_b^2}a'^2\frac{a''}{a} - 4\alpha\frac{\langle \dot{a} \rangle^2}{a^2}a'^2 = (r_b^2 + r_br_c)\langle \dot{a} \rangle^2 - \frac{1}{3}(r_b\rho a^2 + r_b^2\Lambda a^2), \quad (12)$$

and

$$\left[(1 + 4\alpha\frac{\langle \dot{a} \rangle^2}{a^2})\frac{(a^2)'}{2} \right]' - \frac{4\alpha}{3r_b^2}\left[a^2\left(\frac{a'}{a}\right)^3 \right]' + 4\alpha\frac{\langle \dot{a} \rangle^2}{a^2}a'^2 - \frac{4\alpha}{3r_b^2}a^2\left(\frac{a'}{a}\right)^4 = (r_b^2 + r_br_c)\langle \dot{a} \rangle^2 - \frac{1}{3}(r_b\rho a^2 + r_b^2\Lambda a^2). \quad (13)$$

To deal with the complexity contributed by GB effect, we impose the condition

$$a'^2 \ll a \|a''\|, \quad (14)$$

where $\|a''\|$ denotes the absolute value of a'' . It is known that, in the limit of thin brane, the profiles for a'' blow up as delta function [28], hence we expect that the imposed condition (14) could be valid, at least, in the brane with small (but finite) thickness. Thus the third term on l.h.s. in Eq. (12) and the third and fourth terms on the l.h.s. in Eq. (13) can be neglected, which results in

$$(1 + 4\alpha\frac{\langle \dot{a} \rangle^2}{a^2})\frac{(a^2)''}{2} - \frac{4\alpha}{r_b^2}a'^2\frac{a''}{a} = (r_b^2 + r_br_c)\langle \dot{a} \rangle^2 - \frac{1}{3}(r_b\rho a^2 + r_b^2\Lambda a^2), \quad (15)$$

and

$$\left[\left(1 + 4\alpha \frac{\langle \dot{a} \rangle^2}{a^2} \right) \frac{(a^2)'}{2} \right]' - \frac{4\alpha}{3r_b^2} \left[a^2 \left(\frac{a'}{a} \right)^3 \right]' = (r_b^2 + r_b r_c) \langle \dot{a} \rangle^2 - \frac{1}{3} (r_b \rho a^2 + r_b^2 \Lambda a^2). \quad (16)$$

When GB effect disappears, the above two equations can also be obtained even if the condition (14) is released.

Under Z_2 symmetry, integrating Eq. (16) over the brane, one can obtain the boundary condition

$$\left[1 + \frac{8\alpha}{3} \frac{\langle \dot{a} \rangle^2}{a|_{\frac{1}{2}}^2} + \frac{4\alpha}{3} \Phi|_{\frac{1}{2}} \right] (aa')|_{\frac{1}{2}} = \frac{1}{2} (r_b^2 + r_b r_c) \langle \dot{a} \rangle^2 - \frac{1}{6} (r_b \langle \rho a^2 \rangle + r_b^2 \Lambda \langle a^2 \rangle). \quad (17)$$

Solving $a'|_{\frac{1}{2}}$ from that, we have

$$a'|_{\frac{1}{2}} = \frac{1}{\gamma} \frac{\langle a \rangle}{\beta} \left(\frac{1}{2} \zeta r_b^2 H_{\langle a \rangle}^2 - \varepsilon \eta - \frac{1}{6} r_b^2 \Lambda \tilde{\eta} \right), \quad (18)$$

with some dimensionless quantities have been used:

$$\begin{aligned} \varepsilon &\equiv \frac{1}{6} r_b \langle \rho \rangle \\ \gamma &\equiv 1 + \frac{8\alpha}{3\beta^2} H_{\langle a \rangle}^2 + \frac{4\alpha}{3} \Phi|_{\frac{1}{2}} \\ \eta &\equiv \frac{\langle \rho a^2 \rangle}{\langle \rho \rangle \langle a \rangle^2}, \quad \tilde{\eta} \equiv \frac{\langle a^2 \rangle}{\langle a \rangle^2}, \quad \beta \equiv \frac{a|_{1/2}}{\langle a \rangle}, \quad \zeta = \left(1 + \frac{r_c}{r_b} \right). \end{aligned}$$

Whereas ε characterizes the thickness of the brane, the quantities $\beta, \eta, \tilde{\eta}$ characterize the inhomogeneity of the brane along the fifth dimension (in the case of a homogeneous brane, one has $\beta = \eta = \tilde{\eta} = 1$). The quantities γ and ζ embody the GB effect and induced gravity effect, respectively (in the case of RS thick brane, one has $\gamma = \zeta = 1$).

The 55 component of the field equation (2) can be rewritten as

$$\dot{F} = \frac{2}{3} a^3 \dot{a} P_T$$

where

$$F = a^4 \left(\Phi + 2\alpha \Phi^2 - \frac{1}{6} \Lambda \right).$$

Imposing the boundary condition $P_T(\pm \frac{1}{2}) = 0$, one has $\dot{F}(\pm \frac{1}{2}) = 0$, which gives, after time integration,

$$r_b^2 H_{\langle a \rangle}^2 = \frac{a'^2|_{1/2}}{\langle a \rangle^2} + \Phi|_{\frac{1}{2}} r_b^2 \beta^2, \quad (19)$$

where

$$\Phi|_{\frac{1}{2}} = \frac{-1 \pm \sqrt{1 + \frac{4}{3}\alpha \left(\Lambda + \frac{C}{\beta^2 \langle a \rangle^4} \right)}}{4\alpha}. \quad (20)$$

Hereafter, we will select the above branch of (20) for correct limit in $\alpha \rightarrow 0$,

$$\Phi|_{\frac{1}{2}} = \frac{1}{6}\Lambda,$$

and omit the dark radiation term $C = 0$, since in this paper we only care of AdS₅ bulk. The AdS₅ length scale associated to the (negative) cosmological constant in the bulk is defined by

$$l_\Lambda \equiv \sqrt{-\frac{6}{\Lambda}}.$$

Substituting $a'^2|_{1/2}$ (18) into Eq. (19), one has

$$\beta^2 \gamma^2 r_b^2 H_{\langle a \rangle}^2 = \left(\frac{1}{2} \zeta r_b^2 H_{\langle a \rangle}^2 - \varepsilon \eta + \frac{r_b^2}{l_\Lambda^2} \tilde{\eta} \right)^2 + \Phi|_{\frac{1}{2}} r_b^2 \beta^4 \gamma^2.$$

Then the generalized Friedmann equation can be obtained

$$H_{\langle a \rangle}^2 = \frac{2}{r_b^2} \left(\frac{\beta^2 \gamma^2}{\zeta^2} + \frac{\varepsilon \eta}{\zeta} - \frac{r_b^2 \tilde{\eta}}{l_\Lambda^2 \zeta} \right) \left[1 \pm \sqrt{1 - \frac{(\varepsilon \eta - \tilde{\eta} r_b^2 / l_\Lambda^2)^2 + \gamma^2 \Phi|_{\frac{1}{2}} r_b^2 \beta^4}{\zeta^2 \left(\frac{\beta^2 \gamma^2}{\zeta^2} + \frac{\varepsilon \eta}{\zeta} - \frac{r_b^2 \tilde{\eta}}{l_\Lambda^2 \zeta} \right)^2}} \right]. \quad (21)$$

Hereafter, we will select the below branch for correct thin brane limit in $r_b \rightarrow 0$. This is the generalized Friedmann equation governing the cosmological evolution inside the thick brane, in the presence of two curvature correction terms. It is important to notice that this equation should be regraded as an implicit equation for $H_{\langle a \rangle}$ because $\beta, \eta, \tilde{\eta}, \gamma$ can depend on $H_{\langle a \rangle}$.

It is instructive to consider two opposite limits of small and large r_b . In the limit $r_b \rightarrow 0$, Eq. (21) reduces to

$$H_{\langle a \rangle}^2 = \frac{2\beta^2 \gamma^2}{r_c^2} + \frac{\langle \rho \rangle \eta}{3r_c} - \frac{1}{\sqrt{3}r_c} \sqrt{\frac{12\beta^4 \gamma^4}{r_c^2} + \frac{4\beta^2 \gamma^2 \langle \rho \rangle \eta}{r_c} - 12\beta^4 \gamma^2 \Phi|_{\frac{1}{2}}},$$

where the inhomogeneity parameters $\beta, \eta, \tilde{\eta}$ tend to 1 in this limit. One can find that it exactly recovers the junction condition of thin brane cosmology with curvature corrections [20]

$$4\gamma^2 \left(H_{\langle a \rangle}^2 - \Phi|_{\frac{1}{2}} \right) = \left(r_c H_{\langle a \rangle}^2 - \frac{\langle \rho \rangle}{3} \right)^2. \quad (22)$$

It is a cubic equation of $H_{\langle a \rangle}^2$ and we do not write its expatiatory solution clearly (one can find the exact solution in [20]). When GB effect disappears, we can discuss the opposite limit $r_b \rightarrow \infty$ and simplify the Eq. (21) as

$$H_{\langle a \rangle}^2 = \frac{\langle \rho \rangle \eta}{3r_b} - \frac{2\tilde{\eta}}{l_\Lambda^2} + \frac{2r_c}{r_b l_\Lambda^2} + \frac{2\beta}{r_b} \sqrt{\frac{-2\tilde{\eta}^2}{l_\Lambda^2} + \frac{\beta^2}{l_\Lambda^2}},$$

where terms to $O(r_b^{-1})$ have been kept, assuming that the inhomogeneity parameters remain bounded as $r_b \rightarrow \infty$. In the case of a Minkowski bulk, $\Lambda = 0$, one obtains

$$H_{\langle a \rangle}^2 = \frac{\langle \rho \rangle \eta}{3r_b} \quad (23)$$

which corresponds to the standard Friedmann equation. It is interesting to note that the induced gravity correction does not affect the Friedmann equation and the limit $r_b \rightarrow \infty$ allows the possibility of a homogeneous brane along the fifth dimension $\eta = 1$, in agreement with the usual Kaluza-Klein picture.

At the end of this section, we now consider the conservation of matter on thick brane. Since the l.h.s in the field equation (2) is divergence-free, the total energy-momentum tensor is conserved in the bulk

$$\nabla_A T_B^A|_{total} = 0. \quad (24)$$

Using the FRW metric (5), the zero component of Eq. (24) yields

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (25)$$

III. THE EFFECTIVE 4D REDUCTIONS OF COSMOLOGICAL EQUATIONS

There are two effective 4D reductions of cosmological equations in RS thick brane, in the low energy limit and in the limit for a small brane thickness [24]. Accordingly, we will reduce the Friedmann equation (21) and the conservation equation (25).

A. Cosmological equations in the low energy limit

In the thin brane cosmology, the brane tensor is adjusted to compensate the bulk cosmological constant, corresponding to the fine-tuning condition of RS model, which can be reexpressed as a relation

$$l_\lambda = l_\Lambda, \quad (26)$$

where the length scale is defined from the brane tension λ as $l_\lambda \equiv 6/\lambda$. In the current theory, we generalize the version proposed in [24] for RS thick brane

$$l_\lambda \equiv \frac{\tilde{\eta}(1-u)}{\beta^2} l_\Lambda$$

to thick brane with GB curvature correction (while the induced gravity correction does not affect the fine-tuning relation)

$$l_\lambda \equiv \frac{\tilde{\eta}(1-u)}{\beta^2 \left(1 + \frac{4}{3}\alpha \Phi|_{\frac{1}{2}}\right)} \tilde{l}_\Lambda, \quad (27)$$

where the dimensionless ratio has been introduced

$$u \equiv \frac{r_b l_\lambda}{l_\Lambda^2} = -\frac{r_b \Lambda}{\lambda}, \quad (28)$$

and

$$\tilde{l}_\Lambda \equiv \left(-\Phi|_{\frac{1}{2}}\right)^{-\frac{1}{2}}.$$

can be referred as corrected AdS₅ length scale. In the limit where r_b and α go to zero, it can be checked that Eq. (27) reduces to RS condition (26), as it should be.

Now we will consider the low energy regime in which the ordinary energy (and pressure) is small with respect to the brane tensor. To do that, we introduce the dimensionless parameters

$$\varepsilon_\lambda \equiv \frac{r_b}{l_\lambda}, \quad \varepsilon_\rho \equiv \frac{\eta \langle \rho_m \rangle}{\lambda},$$

where

$$\eta \equiv \frac{\langle \rho_m a^2 \rangle}{\langle \rho_m \rangle \langle a^2 \rangle}.$$

Considering the low energy limit in the regime defined by $\varepsilon_\rho \ll \min(1, 1/\varepsilon_\lambda)$, and assuming the GB effect is first order small in ε_ρ , one can find that at lowest order in ε_ρ , Eq. (21) can be expanded as

$$\zeta r_b^2 H_{(a)}^2 = \frac{2\varepsilon_\lambda^2 \varepsilon_\rho}{\varepsilon_\lambda + \frac{\tilde{l}_\Lambda}{\zeta l_\lambda}} - \frac{8\alpha \varepsilon_\lambda^2 \varepsilon_\rho}{\zeta^2 \tilde{l}_\Lambda l_\lambda^3 \left(\varepsilon_\lambda + \frac{\tilde{l}_\Lambda}{\zeta l_\lambda}\right)^2},$$

and the quantities $\beta, \eta, \tilde{\eta}$ must be determined at zeroth order in ε_ρ . To obtain an effective reduction, in the limit at zeroth order in ε_ρ , the 00 component of Einstein equations (see Eq. (11))

$$(1 + 4\alpha\Phi) \frac{a''}{ar_b^2} = \Phi - \frac{1}{3} \left[\frac{\rho}{r_b} - \frac{3r_c}{r_b} \frac{\langle \dot{a} \rangle^2}{a^2} + \Lambda \right]$$

reduces to

$$\begin{aligned} \left(\frac{a^2}{\langle a \rangle^2} \right)'' &= 2\zeta r_b^2 H_{\langle a \rangle}^2 - 4\varepsilon_\lambda (1-u) \frac{a^2}{\langle a \rangle^2} \\ &= -4\varepsilon_\lambda (1-u) \frac{a^2}{\langle a \rangle^2}, \end{aligned} \quad (29)$$

where we have neglected the terms $4\alpha\Phi$ and $2\zeta r_b^2 H_{\langle a \rangle}^2$ which are first order in ε_ρ . Restricting ourselves to the case of physical interest $(1-u) > 0$, and integrating Eq. (29) with the boundary condition $a'(y=0) = 0$, we have

$$a = \langle a \rangle [B \cos(ky)]^{1/2}, \quad (30)$$

with

$$k^2 = 4\varepsilon_\lambda (1-u) \quad (31)$$

and

$$B = \langle \cos^{1/2}(ky) \rangle^{-2}. \quad (32)$$

Simultaneously, one can also obtain

$$\begin{aligned} \beta &= [B \cos(k/2)]^{1/2} \\ \tilde{\eta} &= 2 \frac{B}{k} \sin(k/2). \end{aligned} \quad (33)$$

Here we necessarily have $k \prec \pi$, and the scale factor a is always positive throughout the brane, reaching the extremity of the brane at $y = 1/2$. Now we can determine the evolution of ρ_m at lowest order in ε_ρ . Since neither B nor k depends on t , the conservation equation (25) simply yields

$$\dot{\rho}_m + 3H_{\langle a \rangle}(\rho_m + p_m) = 0, \quad (34)$$

from which it follows that, at this order, ρ_m factorizes as $\rho_m(y, t) = f(y) \langle \rho_m \rangle(t)$. Then one gets

$$\eta = \frac{\langle f(y) \cos(ky) \rangle}{\langle f(y) \rangle \langle \cos^{1/2}(ky) \rangle^2}.$$

The effective 4D thick brane cosmology with induced gravity correction and the first order small GB effect in the limit $\varepsilon_\rho \ll \min(1, 1/\varepsilon_\lambda)$ is thus given by

$$H_{\langle a \rangle}^2 = \frac{\eta}{3 \left(\zeta r_b + \tilde{l}_\Lambda \right)} \langle \rho_m \rangle - \frac{4\alpha}{3} \frac{\eta}{\tilde{l}_\Lambda \left(\zeta r_b + \tilde{l}_\Lambda \right)^2} \langle \rho_m \rangle, \quad (35)$$

and

$$\langle \dot{\rho}_m \rangle + 3H_{(a)}(\langle \rho_m \rangle + \langle p_m \rangle) = 0.$$

The thickness effect is embodied by η besides r_b , which, in this limit, are constant. The induced gravity effect is embodied by constant ζ , and the GB effect is embodied by α . Inserting the fine-tuning condition (27) under infinitely thin limit $r_b \rightarrow 0$ into the junction condition (22) for thin braneworld with induced gravity correction and the first order small GB effect at low energy, one can solve

$$H_{(a)}^2 = \frac{\lambda \langle \rho_m \rangle}{3(6 + 3r_c\lambda)} - \frac{4\alpha\lambda^3 \langle \rho_m \rangle}{27(6 + 3r_c\lambda)^2},$$

which can be exactly recovered from the reduced Friedmann equation (35) under infinitely thin limit $r_b \rightarrow 0$. Also, one can easily find that Eq. (35) recovers the reduction of thick RS braneworld when $\zeta = 1$ and $\alpha = 0$ [24].

Let us discuss the condition (14) in current limit. Using fine-tuning condition (27) with Eqs. (31), (32), and (33), we obtain

$$k = 2 \tan^{-1} \sqrt{\frac{-6r_b \Phi|_{1/2} (1 + \frac{4}{3}\alpha \Phi|_{1/2})^2}{\lambda + r_b\Lambda}}.$$

The infinitely thin limit $r_b \rightarrow 0$ corresponds to $k \rightarrow 0$. Using Eq. (30), one can find that the condition (14) now takes the form

$$\frac{2 \sin^2(ky)}{3 + \cos(2ky)} \ll 1,$$

which is naturally satisfied for small but finite brane thickness.

At last, for our later thermodynamic use, we solve $\langle \rho_m \rangle$ clearly from Eq. (35)

$$\langle \rho_m \rangle = \frac{H_{(a)}^2}{\eta} \left[\frac{18}{\lambda} + 3(r_b + r_c) + \frac{4\alpha\lambda}{3} \right]. \quad (36)$$

B. Cosmological equations for a small brane thickness

Next we wish to discuss another limit case. In order to simplify the calculations, we will restrict ourselves to the case where the bulk cosmological constant vanishes, i.e. the space-time outside the brane effectively is a 5D Minkowski space-time. In this case, one has $\Lambda = \Phi|_{\frac{1}{2}} = l_\Lambda^{-2} = \tilde{l}_\Lambda^{-2} = 0$ and Eq. (21) reads

$$H_{(a)}^2 = \frac{2}{\zeta r_b^2} \left(\frac{\beta^2 \gamma^2}{\zeta} + \varepsilon \eta \right) \left[1 - \sqrt{1 - \frac{(\varepsilon \eta)^2}{(\frac{\beta^2 \gamma^2}{\zeta} + \varepsilon \eta)^2}} \right]. \quad (37)$$

In the following, we will consider the limit $\varepsilon \ll 1$. In contrast with previous subsection where the GB effect is assumed to be small, now we assume the induced gravity effect is small $\zeta \lesssim \gamma^2$. At third order in ε , Eq. (37) can be written as

$$\zeta r_b^2 H_{\langle a \rangle}^2 = \frac{(\varepsilon \eta)^2}{\frac{\beta^2 \gamma^2}{\zeta} + \varepsilon \eta}.$$

At this order, β and η must be expressed at first and zeroth order respectively. To determine a perturbatively in ε we will use the following y expansions

$$\begin{aligned} a(t, y) &= \langle a \rangle(t) \sum_{n=0}^{+\infty} \bar{a}_n(t) y^{2n} \equiv \langle a \rangle(t) \Sigma_a(t, y), \\ \rho(t, y) &= \langle \rho \rangle(t) \sum_{n=0}^{+\infty} \bar{\rho}_n(t) y^{2n} \equiv \langle \rho \rangle(t) \Sigma_\rho(t, y). \end{aligned}$$

Inserting these expansions in Eq. (15) with $\Lambda = 0$, one obtains

$$(1 + 4\alpha H_{\langle a \rangle}^2 \frac{1}{\Sigma_a^2}) (\Sigma_a^2)'' - \frac{8\alpha}{r_b^2} (\Sigma_a')^2 \frac{\Sigma_a''}{\Sigma_a} = 2\zeta r_b^2 H_{\langle a \rangle}^2 - 4\varepsilon \Sigma_\rho \Sigma_a^2. \quad (38)$$

Since one needs a at first order in ε , one can neglect the first term on r.h.s in Eq. (38), and consider the constant (in y) component of Eq. (38) only. We therefore get

$$(1 + 4\alpha H_{\langle a \rangle}^2 \frac{1}{\bar{a}_0^2}) 4\bar{a}_0 \bar{a}_1 = -4\varepsilon \bar{\rho}_0 \bar{a}_0^2.$$

\bar{a}_1 can be solved from this equation

$$\bar{a}_1 = -\frac{\varepsilon}{\theta} \bar{\rho}_0 \bar{a}_0, \quad (39)$$

where the dimensionless parameter is

$$\theta = (1 + 4\alpha H_{\langle a \rangle}^2 \frac{1}{\bar{a}_0^2}).$$

Thus, we have

$$a = \langle a \rangle \bar{a}_0 (1 - \frac{\varepsilon}{\theta} \bar{\rho}_0 y^2). \quad (40)$$

In this expression, θ and $\bar{\rho}_0$ must be determined at zeroth order in ε , which yields $\bar{\rho}_0 = 1$ and $\langle \rho \rangle$ as the solution to the usual 4D energy-momentum conservation equation. From Eq. (40) at first order in ε , one has

$$\bar{a}_0 = \frac{1}{1 - \frac{1}{12} \frac{\varepsilon}{\theta}},$$

and

$$\theta = 1 + 4\alpha H_{\langle a \rangle}^2 \left(1 - \frac{1}{6} \frac{\varepsilon}{\theta}\right).$$

Since θ must be determined at zeroth order in ε , one has

$$\theta = 1 + 4\alpha H_{\langle a \rangle}^2. \quad (41)$$

Eq. (40) finally reads

$$a = \langle a \rangle \left[1 - \frac{\varepsilon}{\theta} \left(y^2 - \frac{1}{12} \right) \right], \quad (42)$$

and the inhomogeneity parameters are $\beta = 1 - \frac{1}{6} \frac{\varepsilon}{\theta}$ and $\eta = 1$. Using Eq. (42), one can find that the condition (14) is naturally satisfied since $\varepsilon \ll 1$.

Now, at third order in ε , the quantity $\zeta r_b^2 H_{\langle a \rangle}^2$ reads

$$\zeta r_b^2 H_{\langle a \rangle}^2 = \frac{\varepsilon^2}{\frac{(1 - \frac{1}{3} \frac{\varepsilon}{\theta}) \gamma^2}{\zeta} + \varepsilon}, \quad (43)$$

or, equivalently,

$$H_{\langle a \rangle}^2 = \frac{\frac{\langle \rho \rangle^2}{36}}{\left(1 - \frac{1}{18} \frac{r_b \langle \rho \rangle}{\theta} \right) \gamma^2 + \frac{(r_b + r_c) \langle \rho \rangle}{6}}, \quad (44)$$

where

$$\gamma = 1 + \frac{8\alpha}{3 \left(1 - \frac{1}{18} \frac{r_b \langle \rho \rangle}{\theta} \right)} H_{\langle a \rangle}^2.$$

The effective 4D Friedmann equation (43) and conservation equation

$$\langle \dot{\rho} \rangle + 3H_{\langle a \rangle} (\langle \rho \rangle + \langle p \rangle) = 0 \quad (45)$$

compose the effective reduced 4D cosmological equations on thick brane.

The equation of $\zeta r_b^2 H_{\langle a \rangle}^2$ is fourth order, whose exact solution is expatiatory and we do not give the exact solution. But one can find that the junction condition (22) of thin braneworld with GB correction and small induced gravity correction can be recovered from Eq. (44) under limit $r_b \rightarrow 0$. An interesting case is when α is very big $\alpha H_{\langle a \rangle}^2 > \varepsilon^{-1}$, Eq. (43) can be simplified as

$$\zeta r_b^2 H_{\langle a \rangle}^2 = \frac{\zeta \varepsilon^2}{\left(\frac{8\alpha}{3} H_{\langle a \rangle}^2 \right)^2}$$

or, equivalently,

$$H_{\langle a \rangle}^2 = \left(\frac{\langle \rho \rangle}{4\alpha} \right)^{\frac{2}{3}},$$

which tells us that as GB effect is big, the effect of small thickness of brane (also small induced gravity effect) can be neglected. Moreover, for later thermodynamic use, we write $\langle \rho \rangle$ from Eq. (44) clearly

$$\langle \rho \rangle = \frac{1}{I} \left\{ 9 \left[H_{\langle a \rangle}^2 \theta J + \sqrt{4(3 + 8\alpha)^2 H_{\langle a \rangle}^2 \theta^2 I - H_{\langle a \rangle}^2 J^2} \right] \right\}, \quad (46)$$

with

$$\begin{aligned} I &= 81\theta^2 + 16\alpha H_{\langle a \rangle}^2 (3 + 8\alpha) r_b^2, \\ J &= 39 + 16\alpha H_{\langle a \rangle}^2 (5 - 4\alpha H_{\langle a \rangle}^2) + 27\theta r_c. \end{aligned}$$

We are also interested in $\langle \rho \rangle$ at first order of r_b , r_c and α

$$\langle \rho \rangle = 6H_{\langle a \rangle} + 3H_{\langle a \rangle}^2 r_c + \frac{22}{3} H_{\langle a \rangle}^2 r_b + 16\alpha H_{\langle a \rangle}^3 - \frac{76}{9} \alpha H_{\langle a \rangle}^4 r_b. \quad (47)$$

IV. THE FIRST LAW OF THERMODYNAMICS ON APPARENT HORIZON OF FRW THICK BRANE

Now we try to construct the first law of thermodynamics on apparent horizon of thick brane. To have further understanding about the nature of apparent horizon we rewrite more explicitly, the 4D metric of FRW universe on the brane in the form

$$ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 d\Omega_2^2,$$

where $h_{ab} = \text{diag}(-1, \frac{a^2}{1-ka^2})$, $d\Omega_2^2$ is the 2-dimensional sphere element, and $x^0 = t$, $x^1 = r$, $\tilde{r} = ar$ is the radius of the sphere and a is the scale factor. For simplicity, we consider the flat space $k = 0$ in this paper. It is known that the dynamical apparent horizon of thin brane, the marginally trapped surface with vanishing expansion, is defined as a sphere situated at $r = r_A$ satisfying

$$h^{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0,$$

which can be solved explicitly,

$$r_A = \frac{1}{\sqrt{\frac{\dot{a}^2}{n^2}}}.$$

The sphere has radius

$$\tilde{r}_A \equiv r_A a = \frac{1}{\sqrt{\frac{\dot{a}^2}{n^2 a^2}}}.$$

The associated temperature on the apparent horizon is

$$T \equiv \frac{\kappa}{2\pi}, \quad (48)$$

where κ is the surface gravity

$$\kappa \equiv \frac{1}{\sqrt{-h}} \partial_a \left(\sqrt{-h} h^{ab} \partial_b \tilde{r} \right) \Big|_{r=r_A}.$$

Define T_a^b as the projection of the 4-dimensional energy-momentum tensor T_ν^μ of a perfect fluid matter in the FRW universe in the normal direction of the 2-sphere. We have work density [29]

$$W \equiv -\frac{1}{2} T_a^a,$$

and energy-supply vector

$$\Psi_a \equiv T_a^b \partial_b \tilde{r} + W \partial_a \tilde{r}.$$

Expressing $A = 4\pi\tilde{r}_A^2$ and $V = \frac{4}{3}\pi\tilde{r}_A^3$ as the area and volume of an 3-dimensional space with radius \tilde{r}_A respectively, we can write the total energy flux on the apparent horizon as

$$\nabla E \equiv A\Psi + W\nabla V.$$

Because of the finite thickness of the brane, there is some arbitrariness in the definition of what the effective 4D quantities should be. Following the order to define the radius of apparent horizon, its area and volume, the temperature on it, the projection of the 4-dimensional energy-momentum tensor, the work density done by a change of it and energy-supply vector flow through it, and finally the total energy flux on it, we define their effective 4D quantities as follows. The location of apparent horizon is determined by

$$\langle h^{ab} \partial_a \tilde{r} \partial_b \tilde{r} \rangle = 0,$$

which has the solution

$$|r_A| \equiv \frac{1}{\sqrt{\langle \frac{\dot{a}^2}{n^2} \rangle}} = \frac{1}{\langle \dot{a} \rangle}.$$

The radius of horizon is set as [31]

$$|\tilde{r}_A| \equiv \langle |r_A| a \rangle = \frac{1}{H_{\langle a \rangle}}. \quad (49)$$

The temperature can be obtained through surface gravity

$$|T| \equiv \frac{|\kappa|}{2\pi} \quad (50)$$

where surface gravity is defined as

$$|\kappa| \equiv \left\langle \frac{1}{\sqrt{-h}} \partial_a \left(\sqrt{-h} h^{ab} \partial_b \tilde{r} \right) \right\rangle \Big|_{r=|r_A|}$$

For later convention, we rewrite it

$$|\kappa| = |r_A| \left\langle \frac{-a}{2} \left[\frac{a}{2\dot{a}} \partial_t \left(\frac{\langle \dot{a} \rangle^2}{a^2} \right) + 2 \frac{\langle \dot{a} \rangle^2}{a^2} \right] \right\rangle. \quad (51)$$

The projection of the 4-dimensional energy-momentum tensor is defined as

$$|T_a^b| \equiv \langle T_a^b \rangle,$$

and work density reads

$$|W| \equiv -\frac{1}{2} |T_a^a| = -\frac{1}{2} \langle T_a^a \rangle,$$

then the energy-supply vector is

$$\begin{aligned} |\Psi_a| &\equiv \langle |T_a^b| \partial_b \tilde{r} + |W| \partial_a \tilde{r} \rangle \Big|_{r=|r_A|} \\ &= \left\langle \langle T_a^b \rangle \partial_b \tilde{r} - \frac{1}{2} \langle T_a^a \rangle \partial_a \tilde{r} \right\rangle \Big|_{r=|r_A|}. \end{aligned}$$

Expressing

$$|A| \equiv 4\pi |\tilde{r}_A|^2, \quad |V| \equiv \frac{4}{3}\pi |\tilde{r}_A|^3$$

as the area and volume of an 3-dimensional space with radius $|\tilde{r}_A|$ respectively, the total energy flux on the apparent horizon can be written as

$$|\nabla E| \equiv |A| |\Psi| + |W| \nabla |V|. \quad (52)$$

For our aim, we write $|A| |\Psi|$ explicitly

$$|A| |\Psi| = 4\pi |\tilde{r}_A|^2 \left[-|\tilde{r}_A| H_{\langle a \rangle} (\langle \rho \rangle + \langle p \rangle) dt + \frac{1}{2} (\langle \rho \rangle + \langle p \rangle) d|\tilde{r}_A| \right]. \quad (53)$$

Now we will consider the thermodynamics of two effective 4D reductions.

A. Thermodynamics in the low energy

Inserting the reduced scale factor a (30), we read the surface gravity (51) on the horizon (49) as

$$\begin{aligned} |\kappa| &= - \left\langle \frac{1}{B \cos(ky)} \right\rangle \frac{|\tilde{r}_A|}{2} \left[\frac{1}{2H_{\langle a \rangle}} \partial_t (H_{\langle a \rangle}^2) + 2 (H_{\langle a \rangle}^2) \right] \\ &= -\frac{1}{B} \left\langle \frac{1}{\cos(ky)} \right\rangle \frac{1}{|\tilde{r}_A|} \left[1 - \frac{\partial_t |\tilde{r}_A|}{2H_{\langle a \rangle} |\tilde{r}_A|} \right], \end{aligned}$$

Using the energy conservation equation (34), we can further draw the surface gravity as

$$|\kappa| = -\frac{1}{B} \left\langle \frac{1}{\cos(ky)} \right\rangle \frac{1}{|\tilde{r}_A|} \left[1 + \frac{3(\langle \rho_m \rangle + \langle p_m \rangle) \partial_t |\tilde{r}_A|}{2 \langle \dot{\rho}_m \rangle |\tilde{r}_A|} \right], \quad (54)$$

and change $|A||\Psi|$ (53) into

$$\begin{aligned} |A||\Psi| &= 4\pi |\tilde{r}_A|^2 \left[|\tilde{r}_A| \frac{\langle \dot{\rho}_m \rangle}{3} dt + \frac{1}{2} (\langle \rho_m \rangle + \langle p_m \rangle) d|\tilde{r}_A| \right] \\ &= \frac{4\pi |\tilde{r}_A|^3}{3} d\langle \rho_m \rangle \left[1 + \frac{3(\langle \rho_m \rangle + \langle p_m \rangle) \partial_t |\tilde{r}_A|}{2 \langle \dot{\rho}_m \rangle |\tilde{r}_A|} \right]. \end{aligned} \quad (55)$$

To build up the first law of thermodynamics, we define entropy

$$\begin{aligned} S &\equiv \int \frac{|A||\Psi|}{T} = \int \frac{2\pi |A||\Psi|}{\kappa} \\ &= - \int \frac{1}{\left\langle \frac{1}{B \cos(ky)} \right\rangle} \frac{8\pi^2 |\tilde{r}_A|^4}{3} d\langle \rho_m \rangle, \end{aligned} \quad (56)$$

where the last equality has been obtained using Eqs. (54) and (55). Substituting $\langle \rho_m \rangle$ (36) and integrating Eq. (56), the entropy can be obtained

$$S = \frac{8\pi^2 B}{\eta \left\langle \frac{1}{\cos(ky)} \right\rangle} \left[\left(\frac{6}{\lambda} |\tilde{r}_A|^2 + r_b |\tilde{r}_A|^2 + r_c |\tilde{r}_A|^2 \right) + \frac{32}{3} \alpha \lambda |\tilde{r}_A|^2 \right]. \quad (57)$$

The physical meaning of this expression is clear: The three terms in parenthesis are contributed by the pure thin RS brane effect, the thick brane effect, and the induced gravity effect, respectively. The remained term describes the first order correction of GB effect. All terms are affected by brane thickness through $B/\langle \eta/\cos(ky) \rangle$.

At last, it should be pointed out that, using the obtained entropy expression (57) and Eqs. (34), (50), (52), (54), and (55), one can construct the first law of thermodynamics

$$|dE| = |T| dS + |W| d|V|.$$

B. Thermodynamics for a small brane thickness

Next we are going to construct the first law in the limit for a small brane thickness by a similar procedure of above subsection. Inserting the reduced scale factor a (42), we obtain the surface gravity (51)

$$\begin{aligned} |\kappa| &= -\frac{|\tilde{r}_A|}{2} \left[\frac{\langle \dot{a} \rangle^2}{\langle a \rangle^2} + \frac{\langle \ddot{a} \rangle}{\langle a \rangle} \right] + \langle 1 - 12y^2 \rangle O(\varepsilon) + O(\varepsilon^2) \\ &= -\frac{|\tilde{r}_A|}{2} \left[\frac{1}{2H_{\langle a \rangle}} \partial_t (H_{\langle a \rangle}^2) + 2 (H_{\langle a \rangle}^2) \right] + O(\varepsilon^2). \end{aligned} \quad (58)$$

Noticing that a is first order in ε , the surface gravity should also be read at first order in ε

$$\begin{aligned} |\kappa| &= -\frac{|\tilde{r}_A|}{2} \left[\frac{1}{2H_{\langle a \rangle}} \partial_t (H_{\langle a \rangle}^2) + 2 (H_{\langle a \rangle}^2) \right] \\ &= -\frac{1}{|\tilde{r}_A|} \left[1 - \frac{\partial_t |\tilde{r}_A|}{2H_{\langle a \rangle} |\tilde{r}_A|} \right], \end{aligned}$$

where the definition of horizon (49) has been used. Using the energy conservation equation (45), we can further read the surface gravity as

$$|\kappa| = -\frac{1}{|\tilde{r}_A|} \left[1 + \frac{3 (\langle \rho_m \rangle + \langle p_m \rangle) \partial_t |\tilde{r}_A|}{2 \langle \dot{\rho}_m \rangle |\tilde{r}_A|} \right], \quad (59)$$

and write $|A| |\Psi|$ (53) as

$$\begin{aligned} |A| |\Psi| &= 4\pi |\tilde{r}_A|^2 \left[|\tilde{r}_A| \frac{\langle \dot{\rho}_m \rangle}{3} dt + \frac{1}{2} (\langle \rho_m \rangle + \langle p_m \rangle) d|\tilde{r}_A| \right] \\ &= \frac{4\pi |\tilde{r}_A|^3}{3} d\langle \rho_m \rangle \left[1 + \frac{3 (\langle \rho_m \rangle + \langle p_m \rangle) \partial_t |\tilde{r}_A|}{2 \langle \dot{\rho}_m \rangle |\tilde{r}_A|} \right]. \end{aligned} \quad (60)$$

To build up the first law of thermodynamics, we define entropy

$$S \equiv \int \frac{|A| |\Psi|}{T} = \int \frac{2\pi |A| |\Psi|}{\kappa} \quad (61)$$

$$= - \int \frac{8\pi^2 |\tilde{r}_A|^4}{3} d\langle \rho_m \rangle, \quad (62)$$

where the last equality has been got using Eqs. (59) and (60). Since $\langle \rho_m \rangle$ (46) is a function of \tilde{r}_A , we can extract the entropy in principle by integrating Eq. (62). However, the exact expression is expatiatory and we only give the entropy at first order α , r_b , r_c through Eq. (47),

$$S = \frac{8\pi^2}{3} \left[\left(2 |\tilde{r}_A|^3 + \frac{22}{3} r_b |\tilde{r}_A|^2 + 3r_c |\tilde{r}_A|^2 \right) + 48\alpha |\tilde{r}_A| - \frac{304}{9} \alpha \log(|\tilde{r}_A|) r_b \right]. \quad (63)$$

The physical meaning of this expression is also clear: The three terms in parenthesis are contributed by the pure thin RS brane effect, the thick brane effect, and the induced gravity effect, respectively. The remained two terms describe the first order GB effect, and the corresponding thickness correction, respectively. Moreover, using the entropy expression (62) (or its first order approximation (63)) and Eqs. (45) (or (47)), (50), (52), (59), and (60), one can construct the first law of thermodynamics

$$|dE| = |T| dS + |W| d|V|.$$

Furthermore, comparing the entropy expressions (57) and (63), we note following interesting features. First, all terms contribute the entropy expression (57) as the form of the Bekenstein-Hawking entropy of the 4D Einstein gravity ($\sim |\tilde{r}_A|^2$). While in entropy expression (63), the pure thin brane effect (the first term of (63)) and the GB effect (the fourth term of (63)) embody the bulk effect [11, 12]. This means that the RS fine-tuning relation (27) and the low energy limit localize the gravity on the brane, whatever the brane has thickness and curvature corrections. Second, the induced gravity effect (the third terms of parenthesis in (57) and in (63)) and the small thick brane effect (the second terms of parenthesis in (63)) also contribute the entropy expressions as the 4D Bekenstein-Hawking entropy. It implies that both effects are localized on four dimensions.

V. CONCLUSION AND DISCUSSION

In this paper, we have generalized the thick RS braneworld scenario presented in [24] in the presence of a 4D scalar curvature from induced gravity on the brane, and a 5D GB curvature term. We have obtained the generalized Friedmann equation governing the cosmological evolution inside the thick brane. As discussed in RS brane world [24], it is a very instructive because the generalized equation interpolates between the familiar 5D GB gravity and the more attractive DGP brane gravity with GB correction. This can be seen clearly from the Eq. (22) for DGP brane cosmology with GB correction when the brane thickness is infinitely thin. For infinitely thick brane, the Kaluza-Klein picture where matter is homogeneously distributed over the extra-dimension and the ordinary Friedmann equations are recovered, is not affected by the induced gravity correction indeed, see Eq. (23). We hope that our generalized Friedmann equation will help in the intuitive understanding of the unconventional DGP brane gravity with GB correction, and clarify the delimitations of the various regimes where different Friedmann equations have to be applied. Moreover, we have proposed the generalized version of the thick RS fine-tuning condition between the bulk cosmological constant and the brane tension. In our case, this cancellation condition depends explicitly on curvature correction besides the brane thickness and yields back the familiar condition without the curvature correction and in the thin brane limit. Using this fine-tuning condition and assuming the cosmological matter content of the brane is small with respect to its tensor, we have got the effective 4D reduced cosmological equations. It

should be noticed that the GB effect is assumed to be small. In the limit where the brane thickness is small, we have found another effective reduced cosmological equations, whereas the induced gravity effect is assumed to be small. Whether the restriction of small curvature corrections can be released is worth to be further explored.

Furthermore, we have studied the thermodynamics of generalized thick braneworld. Similar to the definition of 4D cosmological quantities, the 4D thermodynamic quantities are also defined by integrating over fifth coordinate. It has been shown that two effective 4D reductions of the generalized Friedmann equation can be written directly in the form of the first law of thermodynamics on the apparent horizon. We stress that this result is not intuitive because we do not know previously the reduced Friedmann equations and reduced effective 4D thermodynamics brane quantities. In particular, one should notice that we have omitted the higher-order terms in the reduced thermodynamics brane quantities (58), otherwise we can not extract the entropy from the first law of thermodynamics, see Eq. (61). This result strongly shows that the connection between gravity and thermodynamics is not an accident result. For the generalized Friedmann equation (21) and the conservation equation (25), however, we did not obtain the thermodynamics accordingly. In fact, in [24], it has been shown that if the brane thickness is not small enough, the effective 4D cosmological equations can be obtained but in the price of introducing auxiliary quantities. Observing the Eqs. (56) (62), one can find that the reason why we can extract the entropy expressions is that $\langle \rho_m \rangle$ (36) (46) are the functions of \tilde{r}_A only. If the Friedmann equations have auxiliary quantities, $\langle \rho_m \rangle$ will depend on them and then we can not extract the entropy in general by integrating Eqs. (56) (62). This is a sign that the full 5D description is more adequate, and the 4D thermodynamics can not reflect the gravity of five dimensions. On the other hand, we can speculate that thermodynamics can not reflect the gravity if which has some auxiliary quantities. We emphasize that our study is restricted on equilibrium thermodynamics, whether the generalized Friedmann equations may be described by non-equilibrium thermodynamics is not clear in present work. At last, we would like to point out that our speculation is consonant with the result that no equilibrium thermodynamics is constructed for nonlinear gravity and scalar-tensor gravity, because there are nontrivial auxiliary quantities df/dR [7, 8] and $F(\Phi)$ [9] for nonlinear $f(R)$ gravity and scalar-tensor gravity respectively. We will further investigate whether the auxiliary quantities are essential in the relation between the gravity and thermodynamics in other work [30].

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- [31] Remembering that the effective reductions of Friedmann equation in previous section need

the quantity $\zeta r_b^2 H_{(a)}^2$ is very small, here we like to note that the reductions are carried out when the horizon is very big than brane thickness and crossover scale.